HOMOGENEOUS SPIN RIEMANNIAN MANIFOLDS WITH THE SIMPLEST DIRAC OPERATOR

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Abstract. We show the existence of nonsymmetric homogeneous spin Riemannian manifolds whose Dirac operator is like that on a Riemannian symmetric spin space. Such manifolds are exactly the homogeneous spin Riemannian manifolds (M, g) which are traceless cyclic with respect to some quotient expression $M = G/K$ and reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Using transversally symmetric fibrations of noncompact type, we give a list of them.

Keywords: Dirac operator, homogeneous spin Riemannian manifolds, traceless cyclic homogeneous Riemannian manifolds, Riemannian symmetric spaces.

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1. Introduction

The Dirac operator D on a Riemannian symmetric spin space $(M = G/K, g)$, (G, K) being a Riemannian symmetric pair, may be written (cf. [10, pp. 179–180], [11, p. 230]) as

$$D\psi = \sum_{i=1}^{n} X_i \cdot X_i(\psi), \tag{1.1}$$

where $(X_1, \dots, X_n)$ is a positively oriented orthonormal basis of the -1-eigenspace $\mathfrak{m}$ of the involutive automorphism on the Lie algebra $\mathfrak{g}$ of G determined by (G, K).

Actually, Ikeda [10] gave a general formula for $M = G/K$ being a homogeneous Riemannian manifold, with G unimodular and K compact. In turn, Bär [2, Theorem 1] extended formula (1.1) for G being not necessarily unimodular.

The main purpose of this paper is to show the existence of nonsymmetric homogeneous spin Riemannian manifolds $(M = G/K, g)$, with G semisimple and K connected, whose Dirac operator is expressed as in formula (1.1), and to give a list of them.

To this end, we recall that a homogeneous Riemannian manifold (M, g) is said [6] to be *cyclic* if there exists a quotient expression $M = G/K$ and a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ satisfying

$$\underset{XYZ}{\mathfrak{S}} \langle [X, Y]_{\mathfrak{m}}, Z\rangle = 0, \qquad X, Y, Z \in \mathfrak{m}, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ induced by g. If moreover G is unimodular, (M, g) is said to be *traceless cyclic*.

The study of cyclic and traceless cyclic homogeneous Riemannian manifolds was started by Tricerri and Vanhecke [14] and Kowalski and Tricerri [12]. In the latter

paper, the classification of simply connected traceless cyclic homogeneous Riemannian manifolds of dimension $\leqslant 4$ was given. The present authors extended it to the cyclic case in [6]. The first examples which are not cyclic metric Lie groups [5] do appear in dimension four.

Using the mentioned Bär expression of the Dirac operator on homogeneous spin Riemannian manifolds, we prove the following result.

Theorem 1.1. *A homogeneous spin Riemannian manifold* $(M = G/K, g)$ *has Dirac operator like that on a Riemannian symmetric spin space if and only if it is traceless cyclic.*

Theorem 1.1 yields us to view, in spin geometry, traceless cyclic homogeneous spin Riemannian manifolds as the simplest manifolds after Riemannian symmetric spin spaces.

In the compact case, the expression (1.1) for the Dirac operator on a homogeneous spin Riemannian manifold characterizes the class of Riemannian symmetric spaces. Specifically, we have

Theorem 1.2. *Let* $M = G/K$ *be a homogeneous spin Riemannian manifold with* G *semisimple compact and* K *connected. Then it has Dirac operator like that on a Riemannian symmetric spin space if and only if* (G, K) *is a Riemannian symmetric pair.*

In the noncompact case, we find a large class of nonsymmetric homogeneous spin Riemannian manifolds whose Dirac operator admits an expression like (1.1) with respect to a suitable reductive decomposition. Let G be a connected Lie group and let K and L be closed subgroups of G such that $K \subset L \subset G$. Consider the natural projection $\pi\colon M = G/K \to N = G/L$, $gK \mapsto gL$. Then π gives the homogeneous fibration

$$F = L/K \to M = G/K \xrightarrow{\pi} N = G/L.$$

We say that π is a *transversally symmetric fibration* [8] if (G, L) is a Riemannian symmetric pair. Moreover, π is said to be of *compact type, noncompact type* or *Euclidean type* according to the type of (G, L). We will focus on the noncompact type. Let $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ be a Cartan decomposition, where $\mathfrak{l}$ is the Lie algebra of L and let B be the Killing form of G. We assume that K is compact and connected. Then we have B-orthogonal reductive decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{f}$, where $\mathfrak{m} = \mathfrak{f} \oplus \mathfrak{p}$. A G-invariant metric on M defined by an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{f} \oplus \mathfrak{p}$ is an orthogonal decomposition, is said to be *adapted* to the fibration. Due to Theorem 1.1, we seek for adapted (traceless) cyclic metrics.

The triples (G, L, K) of Lie groups where G/K is a nonsymmetric cyclic homogeneous Riemannian manifold, with K connected, fibering over an irreducible Riemannian symmetric space G/L (necessarily of noncompact type) and with isotropy-irreducible fibre type L/K, are listed, for G a classical simple Lie group, in Tables 1 and 2. For G simple exceptional, the corresponding triples of Lie algebras $(\mathfrak{g}, \mathfrak{l}, \mathfrak{k})$ are given in Table 3.

For transversally symmetric fibrations of noncompact type, we show that there exists a one-to-one correspondence between the sets of homogeneous spin Riemannian structures on the total space M of the fibration π and on the fibre type F. Concretely, we prove the following

Theorem 1.3. *Let $\pi\colon M = G/K \to N = G/L$ be a transversally symmetric fibration of noncompact type. Let $(\widetilde{G}, \widetilde{\pi})$ be the universal covering of G and $\widetilde{L} = \widetilde{\pi}^{-1}(L)$, $\widetilde{K} = \widetilde{\pi}^{-1}(K)$. The total space M, with a metric adapted to the fibration $M = \widetilde{G}/\widetilde{K} \to \widetilde{G}/\widetilde{L}$, is a homogeneous spin Riemannian manifold with adapted (see* Definition 2.1*) quotient $\widetilde{G}/\widetilde{K}$ if and only if the fibre type F, with adapted quotient $\widetilde{L}/\widetilde{K}$, is also a homogeneous spin Riemannian manifold. Moreover, the Dirac operator on M is like that on a Riemannian symmetric spin space if and only if the pair (L, K) is associated with an orthogonal symmetric Lie algebra.*

As a direct consequence, if F is simply connected in Theorem 1.3, the (simply connected) homogeneous spin Riemannian manifold $(M = G/K, g)$ has Dirac operator like that on a Riemannian symmetric spin space if and only if $F = L/K$ is a (compact) Riemannian symmetric spin space.

We recall in Table 4 the Cahen and Gutt [4, Theorems 8 and 12] classification of compact simply connected Riemannian symmetric spin spaces L/K with L simple, with a few minor changes. Then, using Theorem 1.3 and Proposition 4.1, we give many examples of nonsymmetric homogeneous spin Riemannian manifolds G/K having Dirac operator like that on a Riemannian symmetric spin space, in Tables 1, 2 and (through the corresponding pairs $(\mathfrak{g}, \mathfrak{k})$) in Table 3.

The paper is organized as follows. In Section 2 the notion of homogeneous spin Riemannian manifold is recalled and, after giving Bär's expression of the Dirac operator on homogeneous spin Riemannian manifolds, we prove Theorem 1.1. In Section 3, we study nonsymmetric cyclic homogeneous Riemannian manifolds $M = G/K$ with G semisimple. If G is compact and K connected, from Theorem 1.1 above and Proposition 3.2, Theorem 1.2 follows. Then we suppose that G is not compact and K is a connected compact subgroup of G (see Proposition 3.3) and we obtain Tables 1, 2 and 3 (up to the fifth column in Tables 1 and 3, and up to the fourth column in Table 2). In Section 4, Theorem 1.3 is proven, so having examples of nonsymmetric homogeneous spin Riemannian manifolds G/K having Dirac operator like that on a Riemannian symmetric spin space. In Section 5, we give a detailed example, checking that the nonsymmetric manifold $\widetilde{SL(2,\mathbb{R})}$, with each metric of a biparametric family of traceless cyclic metrics, has Dirac operator like that on a Riemannian symmetric spin space.

2. The Dirac operator on traceless cyclic homogeneous spin Riemannian manifolds

Let (M, g) be an n-dimensional spin Riemannian manifold and let $\mathrm{Spin}(n) \hookrightarrow P \to M$ be a fixed spin structure on M. Denote by ρ the basic spin representation of $\mathrm{Spin}(n)$. Let $\Sigma(M)$ be the corresponding spinor bundle over M associated with P, that is,

$$\Sigma(M) = P \times_\rho \Delta,$$

$\Delta = \mathbb{C}^{2^{[n/2]}}$ being the representation space of ρ. Spinor fields are defined as sections $\psi\colon M \to \Sigma(M)$ or, equivalently, as maps $P \to \Delta$ which are equivariant with respect to the action of $\mathrm{Spin}(n)$, that is, $\psi(za) = \rho(a^{-1})\psi(z)$, $z \in P$, $a \in \mathrm{Spin}(n)$.

The Levi-Civita connection ∇ on (M, g) naturally induces [13, Chapter II, Theorem 4.14] a connection ∇^Σ on $\Sigma(M)$, which may be described [1, p. 224] as follows. Let $(e_1, \dots, e_n)$ be a positively oriented local orthonormal frame defined on a connected open subset $U \subset M$. Then $(e_1, \dots, e_n)$ is a local section of the bundle of

positively oriented orthonormal frames $\mathcal{SO}(M)$. Denote by $\hat{\lambda}$ the projection map $\hat{\lambda}\colon P \to \mathcal{SO}(M)$, and let ℓ be a lift to P such that $\hat{\lambda}\circ\ell = (e_1,\dots,e_n)$. Then ℓ defines a trivialization $U\times\Delta \cong \Sigma(M)|_U$ of $\Sigma(M)$ over U, with respect to which one has the following formula for ∇^Σ,

$$\nabla^\Sigma_{e_i}\psi = e_i(\psi) + \frac{1}{4}\sum_{j,k=1}^{n}\Gamma^k_{ij}e_j\cdot e_k\cdot\psi, \tag{2.1}$$

where $\psi \in \Gamma(\Sigma(M))$, $e_i(\psi)$ denotes differentiation of ψ by e_i, Γ^k_{ij} are the corresponding Christoffel symbols with respect to $(e_1,\dots,e_n)$ and $e_j\cdot e_k\cdot\psi := \rho(e_j)\,\rho(e_k)\psi$.

The Dirac operator $D\colon\Gamma(\Sigma(M))\to\Gamma(\Sigma(M))$ is then defined by

$$D\psi = \sum_{i=1}^{n} e_i\cdot\nabla^\Sigma_{e_i}\psi. \tag{2.2}$$

A connected homogeneous Riemannian manifold (M,g) can be described as a coset manifold G/K, where G is a Lie group, which we suppose to be connected, acting transitively and effectively by isometries on M, K is the isotropy subgroup of G at some point $o\in M$, the origin of G/K, and g is a G-invariant Riemannian metric. Moreover, we can assume that G is a closed subgroup of the full isometry group $I(M,g)$ of (M,g). Then, K is a compact subgroup and G/K admits a reductive decomposition, that is, there is an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{g}$ splits as the vector space direct sum $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$, $\mathfrak{k}$ being the Lie algebra of K.

Next, suppose that $(M = G/K, g)$ is an oriented homogeneous Riemannian manifold with a fixed reductive decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$. By using the identification $T_o(M)\cong\mathfrak{m}$ and taking into account that G is connected, the isotropy representation χ may be expressed as the map $\chi\colon K\to SO(\mathfrak{m})$, $\chi(k)(X)=\mathrm{Ad}_k(X)$, for all $k\in K$ and $X\in\mathfrak{m}$. Then the choice of a positively oriented orthonormal basis $(X_1,\dots,X_n)$ of $\mathfrak{m}$ gives us the identification

$$G\times_\chi SO(\mathfrak{m}) \equiv \mathcal{SO}(M),$$

via the map $[(a,A)]\mapsto(\tau_a(o);(\tau_a)_{*o}AX_1,\dots,(\tau_a)_{*o}AX_n)$. Let $\lambda\colon\mathrm{Spin}(\mathfrak{m})\to SO(\mathfrak{m})$ be the usual two-sheet covering map. If there exists a Lie group homomorphism $\widetilde{\chi}\colon K\to\mathrm{Spin}(\mathfrak{m})$ lifting χ, that is, $\lambda\circ\widetilde{\chi}=\chi$, we can define a spin structure $P_{\widetilde{\chi}}$ on M by

$$P_{\widetilde{\chi}} = G\times_{\widetilde{\chi}}\mathrm{Spin}(\mathfrak{m}),$$

with covering map $\hat{\lambda}\colon P_{\widetilde{\chi}}\to\mathcal{SO}(M)=G\times_\chi SO(\mathfrak{m})$, given by $\hat{\lambda}([(a,A)]=[(a,\lambda(A))]$.

Throughout this paper, we consider homogeneous spin Riemannian manifolds as in the next (well-known) definition.

Definition 2.1. An oriented connected homogeneous Riemannian manifold (M,g) of dimension n is said to be a *homogeneous spin Riemannian manifold* if there exists a quotient expression $M=G/K$ and a reductive decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ such that the isotropy representation χ can be lifted to a Lie group homomorphism $\widetilde{\chi}\colon K\to\mathrm{Spin}(\mathfrak{m})$. The spin structure $\mathrm{Spin}(n)\to P_{\widetilde{\chi}}=G\times_{\widetilde{\chi}}\mathrm{Spin}(\mathfrak{m})\to M$ on M is called a *homogeneous spin structure*. Then G/K (resp., $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$) is said to be a quotient expression (resp., reductive decomposition) *adapted* to the homogeneous spin structure.

A homogeneous Riemannian manifold $(M = G/K, g)$ with G simply connected and admitting a spin structure is a homogeneous spin Riemannian manifold. Concretely, we have the following

Lemma 2.2. [2, Lemma 3] *If G is simply connected, there exists a one-to-one correspondence between the set of spin structures of the oriented manifold $(M = G/K, g)$ and the set of lifts $\widetilde{\chi}$ of χ.*

Consider the universal covering $(\widetilde{G}, \widetilde{\pi})$ of a connected Lie group G. Then $\widetilde{\pi}\colon \widetilde{G} \to G$ is a Lie group homomorphism. Put $\widetilde{K} = \widetilde{\pi}^{-1}(K)$ and denote by $\mathrm{Ad}^{\widetilde{G}}$ the adjoint representation of $\widetilde{G}$.

Lemma 2.3. *Let $(G/K, g)$ be a homogeneous Riemannian manifold with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and G-invariant metric g determined by an $\mathrm{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$. If $\mathfrak{m}$ and $\langle \cdot, \cdot \rangle$ are $\mathrm{Ad}^{\widetilde{G}}(\widetilde{K})$-invariant, then $(G/K, g)$ is naturally isometric to $(\widetilde{G}/\widetilde{K}, \tilde{g})$, where $\tilde{g}$ is the $\widetilde{G}$-invariant metric determined by $\langle \cdot, \cdot \rangle$.*

Proof. Because $\widetilde{G}$ acts transitively on the left on G/K via the mapping $(\tilde{a}, \widetilde{\pi}(\tilde{b})K) \mapsto \widetilde{\pi}(\tilde{a}\tilde{b})K$, for all $\tilde{a}, \tilde{b} \in \widetilde{G}$, we have that the map $j\colon \widetilde{G}/\widetilde{K} \to G/K$, defined by $j(\tilde{a}\widetilde{K}) = \widetilde{\pi}(\tilde{a})K$, is a diffeomorphism. Denote by $\tau_{\widetilde{\pi}(\tilde{a})}$ and $\tilde{\tau}_{\tilde{a}}$ the translations on G/K and $\widetilde{G}/\widetilde{K}$, respectively. Then,

$$\tau_{\widetilde{\pi}(\tilde{a})} \circ j = j \circ \tilde{\tau}_{\tilde{a}}, \quad \text{for all } \tilde{a} \in \widetilde{G}.$$

Denoting by o and $\tilde{o}$ the corresponding origins of G/K and $\widetilde{G}/\widetilde{K}$, the tangent spaces $T_o(G/K)$ and $T_{\tilde{o}}(\widetilde{G}/\widetilde{K})$ are identified with $\mathfrak{m}$ by the isomorphism j_{*o}. Hence, for all $u, v \in T_{\tilde{\tau}_{\tilde{a}}(\tilde{o})}(\widetilde{G}/\widetilde{K})$, we get

$$\begin{aligned}
\tilde{g}_{\tilde{\tau}_{\tilde{a}}(\tilde{o})}(u, v) &= \left\langle (\tilde{\tau}_{\tilde{a}^{-1}})_{*\tilde{\tau}_{\tilde{a}}(\tilde{o})} u, (\tilde{\tau}_{\tilde{a}^{-1}})_{*\tilde{\tau}_{\tilde{a}}(\tilde{o})} v \right\rangle \\
&= \left\langle j_{*\tilde{o}} (\tilde{\tau}_{\tilde{a}^{-1}})_{*\tilde{\tau}_{\tilde{a}}(\tilde{o})} u, j_{*\tilde{o}} (\tilde{\tau}_{\tilde{a}^{-1}})_{*\tilde{\tau}_{\tilde{a}}(\tilde{o})} v \right\rangle \\
&= \left\langle (\tau_{\pi(\tilde{a}^{-1})})_{*\tau_{\pi(\tilde{a})}(\tilde{o})} j_{*\tilde{\tau}_{\tilde{a}}(\tilde{o})} u, (\tau_{\pi(\tilde{a}^{-1})})_{*\tau_{\pi(\tilde{a})}(\tilde{o})} j_{*\tilde{\tau}_{\tilde{a}}(\tilde{o})} v \right\rangle \\
&= g_{\tau_{\pi(\tilde{a})}(o)}(j_{*\tilde{\tau}_{\tilde{a}}(\tilde{o})} u, j_{*\tilde{\tau}_{\tilde{a}}(\tilde{o})} v).
\end{aligned}$$

□

Proposition 2.4. *A simply connected homogeneous Riemannian manifold $(M = G/K, g)$ equipped with a spin structure is a homogeneous spin Riemannian manifold with adapted quotient expression $\widetilde{G}/\widetilde{K}$.*

Proof. It follows from Lemmas 2.2 and 2.3, using that $\widetilde{K}$ is connected. □

Remark 2.5. A (Riemannian) manifold M admits a spin structure if and only if the Stiefel-Whitney classes $w_1(M)$ and $w_2(M)$ vanish. A Riemannian symmetric space M of noncompact type is contractible [11, Chapter XI, Theorem 8.6], so its Stiefel-Whitney classes vanish and then it admits a spin structure. From Proposition 2.4, M is a homogeneous spin Riemannian manifold.

The spinor bundle $\Sigma_{\widetilde{\chi}}(M)$ of the homogeneous spin structure $P_{\widetilde{\chi}}$ is given [2, Lemma 4] by

$$\Sigma_{\widetilde{\chi}}(M) = G \times_{\rho \circ \widetilde{\chi}} \Delta.$$

Hence, spinor fields can be viewed as $\rho \circ \widetilde{\chi}$-equivariant maps $\psi\colon G \to \Delta$.

The spinor connection ∇^Σ induced by the Levi-Civita connection of (M,g) is given [2] by

$$\nabla^\Sigma_{(\tau_g)_{*o}X}\psi = \Big(X(\psi) + \frac{1}{4}\sum_{i,j} c_{ij}(X) X_i \cdot X_j \cdot \psi\Big)(g),$$

for all $X \in \mathfrak{m}$, $g \in G$, where $X(\psi)(g) = (\mathrm{d}/\mathrm{d}t)_{|t=0}(\psi(g \exp tX))$ and

$$c_{ij}(X) = \langle \nabla_{X_i} X^*, X_j\rangle = \frac{1}{2}\{-\langle [X_i, X]_\mathfrak{m}, X_j\rangle + \langle [X_j, X_i]_\mathfrak{m}, X\rangle + \langle [X_j, X]_\mathfrak{m}, X_i\rangle\},$$

where X^* stands for the fundamental vector field associated to X, that is, $X^*_p = (\mathrm{d}/\mathrm{d}t)_{|t=0}((\exp tX)p)$, for all $p \in M$. In particular, if (G,K) is a Riemannian symmetric pair then $\nabla^\Sigma_{(\tau_g)_{*o}X}\psi = X(\psi)$ and the Dirac operator may be written, in terms of a positively oriented orthonormal basis $(X_1, \dots, X_n)$ of $\mathfrak{m}$ (cf. [10]), as in (1.1). For homogeneous spin Riemannian manifolds $(M = G/K, g)$, with adapted reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, the Dirac operator is then given by

$$\begin{aligned} (D\psi)(g) &= \textstyle\sum_k (\tau_g)_{*o}X_k \cdot (\nabla^\Sigma_{(\tau_g)_{*o}X_k}\psi) \\ &= \textstyle\sum_k (X_k \cdot X_k(\psi) + \frac{1}{4}\sum_{i,j} c_{ijk} X_k \cdot X_i \cdot X_j(\psi))(g), \end{aligned}$$

where $c_{ijk} = c_{ij}(X_k)$. Hence, the determination of the coefficients c_{ijk} gives the Bär formula [2, §2], which, using that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and K is compact, can be expressed as

$$\begin{aligned} D\psi = \sum_{i=1}^n X_i \cdot X_i(\psi) + \frac{1}{4}\Big(&\sum_{i<j<k} \Big(\mathop{\mathfrak{S}}_{X_i X_j X_k} \langle [X_i, X_j]_\mathfrak{m}, X_k\rangle \Big) X_i \cdot X_j \cdot X_k \\ &- 2\sum_{i=1}^n (\operatorname{tr} \operatorname{ad}_{X_i}) X_i \Big) \cdot \psi. \end{aligned} \tag{2.3}$$

Remark 2.6. Let $\mathrm{Cl}(\mathfrak{m})$ be the Clifford algebra of the negative form on $\mathfrak{m}$ and let $\mathrm{Cl}_\mathbb{C}(\mathfrak{m})$ be its complexified Clifford algebra. On account of the maps $\mathfrak{m} \hookrightarrow \mathrm{Cl}(\mathfrak{m}) \hookrightarrow \mathrm{Cl}_\mathbb{C}(\mathfrak{m}) \xrightarrow{\rho} \mathrm{End}(\Delta)$, each X_i in (2.3) acts on spinors as an element of $\mathrm{End}(\Delta) = \mathrm{End}(\mathbb{C}^{2^{[n/2]}})$.

Proof of Theorem 1.1. Suppose that $(M = G/K, g)$ is a traceless cyclic homogeneous spin Riemannian manifold with adapted reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then the last two summands in (2.3) vanish and we are left with the expression (1.1), which is the one for a Riemannian symmetric spin space.

Conversely, let $(M = G/K, g)$ be a homogeneous spin Riemannian manifold and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be an adapted reductive decomposition. Suppose that it has Dirac operator like that on a Riemannian symmetric spin space. Then we have, on account of (1.1) and (2.3), that

$$\sum_{i<j<k} \Big(\mathop{\mathfrak{S}}_{X_i X_j X_k} \langle [X_i, X_j]_\mathfrak{m}, X_k\rangle \Big) X_i \cdot X_j \cdot X_k - 2\sum_{i=1}^n (\operatorname{tr} \operatorname{ad}_{X_i}) X_i = 0.$$

Now, if $n \geqslant 3$, then X_i, $X_j \cdot X_k \cdot X_l$, $i = 1, \dots, n$, $1 \leqslant j < k < l \leqslant n$, are linearly independent elements of $\mathrm{Cl}_\mathbb{C}(\mathfrak{m})$, hence each cyclic sum $\mathfrak{S}_{X_i X_j X_k}\langle [X_i X_j]_\mathfrak{m}, X_k\rangle$ and each coefficient $\operatorname{tr} \operatorname{ad}_{X_i}$ is null. It is immediate that $(M = G/K, g)$ is traceless cyclic and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an adapted reductive decomposition.

If $n = 2$, it is immediate from [14, Theorem 3.1] and [6, Definition 4.1] that G/K is cyclic. Now, if M has Dirac operator like that on a Riemannian symmetric

spin space, one has $\operatorname{tr}\operatorname{ad}_X = 0$, for all $X \in \mathfrak{m}$, and hence G is unimodular, then $M = G/K$ is also traceless cyclic. $\square$

3. Cyclic homogeneous Riemannian manifolds G/K with G semisimple

Let $(M = G/K, g)$ be a homogeneous Riemannian manifold with G semisimple and let B be the Killing form of G. Since the isotropy subgroup K is compact, the restriction of B to $\mathfrak{k}$ is negative definite and one gets the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the B-orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Such decomposition is clearly reductive. Then each G-invariant metric on M is uniquely determined by an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$.

Lemma 3.1. *Let $\mathfrak{m} = \mathfrak{f} \oplus \mathfrak{p}$ be a B-orthogonal decomposition of $\mathfrak{m}$ into two $\mathrm{Ad}(K)$-invariant subspaces $\mathfrak{f}$ and $\mathfrak{p}$ ($\mathfrak{f}$ or $\mathfrak{p}$ may be zero). Suppose that the restrictions $B_{|\mathfrak{f}\times\mathfrak{f}}$ and $B_{|\mathfrak{p}\times\mathfrak{p}}$ are negative definite and positive definite, respectively. Then, for any $\mathrm{Ad}(K)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$, making orthogonal the subspaces $\mathfrak{f}$ and $\mathfrak{p}$, there is an orthogonal splitting*

$$\mathfrak{m} = \mathfrak{f}_1 \oplus \cdots \oplus \mathfrak{f}_r \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s, \tag{3.1}$$

where K acts irreducibly on $\mathfrak{f}_i$ and $\mathfrak{p}_j$, for $i = 1, \dots, r$ and $j = 1, \dots, s$, by the adjoint map and $\langle\cdot,\cdot\rangle$ is a diagonal *direct sum with respect to B of the form*

$$\langle\cdot,\cdot\rangle = \sum_{a=1}^{r} \lambda_a B_{|\mathfrak{f}_a\times\mathfrak{f}_a} + \sum_{b=1}^{s} \lambda_{r+b} B_{|\mathfrak{p}_b\times\mathfrak{p}_b}, \tag{3.2}$$

where $\lambda_a < 0$ for $a = 1, \dots, r$, and $\lambda_{r+b} > 0$ for $b = 1, \dots, s$.

Proof. Let Q be the unique symmetric bilinear map on $\mathfrak{m}$ such that

$$\langle x, y\rangle = B(Qx, y), \qquad x, y \in \mathfrak{m}.$$

Using that B is nondegenerate and $\langle\cdot,\cdot\rangle$ and B are both $\mathrm{Ad}(K)$-invariant, one gets that $Q \circ \mathrm{Ad}_k = \mathrm{Ad}_k \circ Q$, for all $k \in K$. Hence, the eigenspaces of Q are $\mathrm{Ad}(K)$-invariant subspaces. They are mutually orthogonal for both $\langle\cdot,\cdot\rangle$ and B. Moreover, on each irreducible $\mathrm{Ad}(K)$-invariant subspace, both $\langle\cdot,\cdot\rangle$ and B must be proportional, with proportionality factor the corresponding eigenvalue of Q. Since $\langle\cdot,\cdot\rangle$ is strictly definite on $\mathfrak{f}$ and on $\mathfrak{p}$, such factors must be different from zero. $\square$

Choose a B-orthogonal basis $\{e_i^a : 1 \leqslant i \leqslant n_a, 1 \leqslant a \leqslant r+s\}$ adapted to the splitting (3.1), where $n_a = \dim \mathfrak{f}_a$, if $a = 1, \dots, r$, and $n_{r+b} = \dim \mathfrak{p}_b$, if $b = 1, \dots, s$. This basis is $\langle\cdot,\cdot\rangle$-orthogonal. Put

$$[e_i^a, e_j^b]_{\mathfrak{m}} = \sum_{\substack{1\leqslant c\leqslant r+s\\ 1\leqslant k\leqslant n_c}} c_{i_a j_b}^{k_c} e_k^c.$$

Then, arguing as in [6, Proposition 8.2], the next result follows.

Proposition 3.2. *The G-invariant metric on M determined by the inner product (3.2) is (traceless) cyclic if and only if*

$$c_{i_a j_b}^{k_c}(\lambda_a + \lambda_b + \lambda_c) = 0, \tag{3.3}$$

where $a, b, c = 1, \dots, r+s$, $1 \leqslant i \leqslant n_a$, $1 \leqslant j \leqslant n_b$, $1 \leqslant k \leqslant n_c$.

Proof of Theorem 1.2. From Proposition 3.2 we have that the pair (G, K) is a Riemannian symmetric pair, since it is associated with an orthogonal symmetric Lie algebra of compact type and K is connected [9, Chapter VII, Exercise 10]. By using now Theorem 1.1, we obtain Theorem 1.2.

We now consider cyclic homogeneous Riemannian manifolds G/K with G semisimple noncompact. More precisely, let $\pi\colon M = G/K \to N = G/L$ be a transversally symmetric fibration of noncompact type, with K compact and connected. Consider the maximal compact subgroup L of G such that $K \subset L$ and let $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ be the Cartan decomposition, where $\mathfrak{l}$ is the Lie algebra of L. Let $\mathfrak{m}$ be the direct sum $\mathfrak{m} = \mathfrak{f} \oplus \mathfrak{p}$, $\mathfrak{f}$ being the B-orthogonal complement to the Lie algebra $\mathfrak{k}$ of K in $\mathfrak{l}$. Then, we have

Proposition 3.3. *The homogeneous manifold $M = G/K$ admits a cyclic metric adapted to the transversally symmetric fibration $\pi\colon M = G/K \to N = G/L$ if and only if the pair (L, K) is associated to an orthogonal symmetric Lie algebra $(\mathfrak{l}, s)$.*

Proof. Because B is strictly definite on $\mathfrak{l}$ ($B < 0$) and on $\mathfrak{p}$ ($B > 0$), there are B-orthogonal decompositions $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$, $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{f}$, $\mathfrak{m} = \mathfrak{f} \oplus \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where

$$[\mathfrak{l}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{f}] \subset \mathfrak{f}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}.$$

We also have $[\mathfrak{f}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{f}, \mathfrak{f}]_{\mathfrak{m}} \subset \mathfrak{f}$. Since (G, L) is a symmetric pair, one has $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l}$, and hence

$$[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{m}} \subset (\mathfrak{k} \oplus \mathfrak{f})_{\mathfrak{m}} = \mathfrak{f}.$$

Suppose first that there exists a cyclic metric on G/K with respect to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ which is defined by an $\mathrm{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ making orthogonal $\mathfrak{f}$ and $\mathfrak{p}$. From Lemma 3.1, $\langle \cdot, \cdot \rangle$ takes the form (3.2). Because $[\mathfrak{f}, \mathfrak{f}]_{\mathfrak{m}} \subset \mathfrak{f}$, equation (3.3) implies that $[\mathfrak{f}, \mathfrak{f}]_{\mathfrak{m}} = 0$, that is, $[\mathfrak{f}, \mathfrak{f}] \subset \mathfrak{k}$. Hence $(\mathfrak{l}, s)$ is an orthogonal symmetric Lie algebra, where s is the involutive automorphism of $\mathfrak{l}$ with eigenspaces $\mathfrak{k}$ and $\mathfrak{f}$ for the eigenvalues $+1$ and -1, respectively and, since L is connected, (L, K) is a pair associated to it.

Conversely, if $(\mathfrak{l}, s)$ is an orthogonal symmetric Lie algebra associated to (L, K), one has $[\mathfrak{f}, \mathfrak{f}] \subset \mathfrak{k}$ and hence $[\mathfrak{f}, \mathfrak{f}]_{\mathfrak{m}} = 0$. So if $\langle \cdot, \cdot \rangle$ is an inner product on $\mathfrak{m}$ we have $\mathfrak{S}_{XYZ} \langle [X, Y]_{\mathfrak{m}}, Z \rangle = 0$ for every $X, Y, Z \in \mathfrak{f}$.

Moreover, since the subspaces $\mathfrak{f}$ and $\mathfrak{p}$ of $\mathfrak{m}$ are invariant by the adjoint representation $\mathrm{Ad}\colon K \to \mathrm{Aut}(\mathfrak{m})$, the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ making $\mathfrak{f}$ and $\mathfrak{p}$ orthogonal and given by

$$\langle \cdot, \cdot \rangle = -2\, B_{|\mathfrak{f} \times \mathfrak{f}} + B_{|\mathfrak{p} \times \mathfrak{p}}$$

is $\mathrm{Ad}(K)$-invariant. Hence, we also have $\mathfrak{S}_{XYZ} \langle [X, Y]_{\mathfrak{m}}, Z \rangle = 0$ if either $X, Y \in \mathfrak{f}$ and $Z \in \mathfrak{p}$ (because $[\mathfrak{f}, \mathfrak{f}]_{\mathfrak{m}} = 0$ and $[\mathfrak{f}, \mathfrak{p}] \subset \mathfrak{p}$) or if $X, Y, Z \in \mathfrak{p}$ (as $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{m}} \subset \mathfrak{f}$).

Finally, if $X, Y \in \mathfrak{p}$ and $Z \in \mathfrak{f}$, using that B is $\mathrm{Ad}(G)$-invariant, one gets

$$\begin{aligned} \mathfrak{S}_{XYZ} \langle [X, Y]_{\mathfrak{m}}, Z \rangle &= -2B([X, Y]_{\mathfrak{m}}, Z) + B([Y, Z]_{\mathfrak{m}}, X) + B([Z, X]_{\mathfrak{m}}, Y) \\ &= -2B([X, Y], Z) + B([Y, Z], X) + B([Z, X], Y) = 0. \qquad \square \end{aligned}$$

Remark 3.4. *Since G/L is contractible, the exact homotopy sequence of the fibration π,*

$$\cdots \to \pi_2(G/L) \to \pi_1(L/K) \to \pi_1(G/K) \to \pi_1(G/L),$$

shows that L/K is simply connected if and only if $M = G/K$ is so.

The fibre F_x through an arbitrary point $x = \tau_a(o)$, for some $a \in G$, is given by $F_x = \tau_a(F)$, where F is the fibre type $F = L/K$. With respect to an adapted metric, the fibres are totally geodesic but the corresponding foliation is not necessarily Riemannian (see [8] for more details). From Proposition 3.3, the existence of adapted cyclic metrics on M implies that F is locally symmetric and, from Remark 3.4, it is globally symmetric if M is simply connected. Moreover, due to the connectedness of K, also F is globally symmetric if the orthogonal symmetric Lie algebra $(\mathfrak{l}, s)$ is of compact type, that is, $\mathfrak{l}$ is semisimple.

In order to obtain Tables 1, 2 and 3, we shall need the following lemma.

Lemma 3.5. *Let $(\mathfrak{l}, s)$ be an orthogonal symmetric Lie algebra with $\mathfrak{l}$ a compact Lie algebra. Let $\mathfrak{k}$ and $\mathfrak{f}$ be the eigenspaces of s for the eigenvalues $+1$ and -1, respectively. Suppose that the dimension of the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ is less than or equal to one. If the Lie algebra $\mathrm{ad}_{\mathfrak{l}}(\mathfrak{k})$ acts irreducibly on $\mathfrak{f}$, $(\mathfrak{l}, s)$ belongs to one of the following classes:*

(i) Type S1 : $(\mathfrak{l}, s)$ *is an irreducible orthogonal symmetric Lie algebra*;
(ii) Type S2 : $\mathfrak{l}$ *is semisimple and there exists a proper ideal $\tilde{\mathfrak{l}}$ of $\mathfrak{l}$ such that the pair $(\tilde{\mathfrak{l}}, \tilde{s})$, where $\tilde{s}$ is the restriction of s to $\tilde{\mathfrak{l}}$, is an irreducible orthogonal symmetric Lie algebra and $\mathfrak{f}$ is the -1-eigenspace of $\tilde{s}$*;
(iii) Type NS0 : $\mathfrak{l}$ *is not semisimple and $(\mathfrak{l}, s)$ is effective. Then $\mathfrak{f} = \mathfrak{z}(\mathfrak{l})$ and so, $(\mathfrak{l}, s)$ is of Euclidean type*;
(iv) Type NS1 : $\mathfrak{l}$ *is not semisimple, $(\mathfrak{l}, s)$ is not effective and $(\mathfrak{l}_-, s_-)$ is an irreducible orthogonal symmetric Lie algebra, where $\mathfrak{l}_-$ is the semisimple ideal $[\mathfrak{l}, \mathfrak{l}]$ of $\mathfrak{l}$*;
(v) Type NS2 : $\mathfrak{l}$ *is not semisimple, $(\mathfrak{l}, s)$ is not effective and there exists a proper ideal $\tilde{\mathfrak{l}}_-$ of $\mathfrak{l}_-$ such that the pair $(\tilde{\mathfrak{l}}_-, \tilde{s}_-)$, where $\tilde{s}_-$ is the restriction of s to $\tilde{\mathfrak{l}}_-$, is an irreducible orthogonal symmetric Lie algebra and $\mathfrak{f}$ is the -1-eigenspace of $\tilde{s}_-$.*

Proof. First, suppose that $\mathfrak{l}$ is semisimple. Then the pair $(\mathfrak{l}, s)$ is an orthogonal symmetric Lie algebra of compact type. When $\mathfrak{k}$ contains no nonzero ideal of $\mathfrak{l}$, $(\mathfrak{l}, s)$ is irreducible, and it is of type S1. Otherwise, let $\mathfrak{u}$ be the maximal ideal of $\mathfrak{l}$ contained in $\mathfrak{k}$ and let $\tilde{\mathfrak{l}}$ be the semisimple compact ideal, orthogonal complement of $\mathfrak{u}$ with respect to B. Because $\mathfrak{u} \subset \mathfrak{k}$ and B is invariant by s, it follows that s preserves $\tilde{\mathfrak{l}}$. Hence, the restriction $\tilde{s}$ of s to $\tilde{\mathfrak{l}}$ is an involutive automorphism of $\tilde{\mathfrak{l}}$. Since the set $\tilde{\mathfrak{k}} = \mathfrak{k} \cap \tilde{\mathfrak{l}}$ of fixed points of $\tilde{s}$ is a compactly embedded subalgebra of $\mathfrak{l}$, we have that $(\tilde{\mathfrak{l}}, \tilde{s})$ is an orthogonal symmetric Lie algebra, which is moreover by construction irreducible and $\mathfrak{f}$ is the -1-eigenspace of $\tilde{s}$. Then, $(\mathfrak{l}, s)$ is of type S2.

Next, suppose that $\mathfrak{l}$ is not semisimple. Then $\mathfrak{l}$ is the direct sum $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_-$, where $\mathfrak{l}_0$ and $\mathfrak{l}_-$ are the ideals $\mathfrak{z}(\mathfrak{l})$ and $[\mathfrak{l}, \mathfrak{l}]$, respectively. These subspaces are invariant under s and orthogonal with respect to the Killing form B of $\mathfrak{l}$. Then $(\mathfrak{l}_0, s_0)$ and $(\mathfrak{l}_-, s_-)$ are orthogonal symmetric Lie algebras, where s_0 and s_- denote the restrictions of s to $\mathfrak{l}_0$ and $\mathfrak{l}_-$, respectively [9, Chapter V, Theorem 1.1]. Let $\mathfrak{l}_0 = \mathfrak{k}_0 \oplus \mathfrak{f}_0$ and $\mathfrak{l}_- = \mathfrak{k}_- \oplus \mathfrak{f}_-$ be the decomposition of $\mathfrak{l}_0$ and $\mathfrak{l}_-$ into the corresponding ± 1-eigenspaces of s_0 and s_-. Then the subspaces $\mathfrak{k}_0$ and $\mathfrak{k}_-$ are ideals in $\mathfrak{k}$, orthogonal with respect to B, and $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_-$. Because the subspaces $\mathfrak{f}_0$ and $\mathfrak{f}_-$ of $\mathfrak{f}$ are $\mathrm{ad}_{\mathfrak{l}}(\mathfrak{k})$-invariant, one of them must be trivial. If $(\mathfrak{l}, s)$ is effective, that is, $(\mathfrak{l}, s)$ is of type NS0, one gets that $\mathfrak{k}_0 = \{0\}$ and $\mathfrak{l}_0 = \mathfrak{f}_0$. Hence, $\mathfrak{f}_-$ is trivial and so, $\mathfrak{l}_0 = \mathfrak{f}$ and $\dim \mathfrak{l}_0 = 1$.

If $(\mathfrak{l}, s)$ is not effective, then $\mathfrak{l}_0 = \mathfrak{k}_0$ and so, $\mathfrak{f}_0 = \{0\}$. Hence, the types NS1 and NS2 for $(\mathfrak{l}, s)$ correspond with the types S1 and S2 for the orthogonal symmetric Lie algebra $(\mathfrak{l}_-, s_-)$ (of compact type), respectively. Finally, the case (v), for the type NS2, is proved using the same proof as for the case (ii). □

Given an irreducible Riemannian symmetric space of noncompact type G/L, where L is a maximal compact subgroup of G, we consider orthogonal symmetric Lie algebras $(\mathfrak{l}, s)$, where $\mathfrak{l}$ is the Lie algebra of L, satisfying the following two conditions:

(i) the algebra $\mathrm{ad}_{\mathfrak{l}}(\mathfrak{k})$ acts irreducibly on $\mathfrak{f}$, where $\mathfrak{k}$ and $\mathfrak{f}$ are the eigenspaces of s for the eigenvalues $+1$ and -1, respectively;
(ii) the connected subgroup K of L, with Lie algebra $\mathfrak{k}$, is closed (this is the case if $\mathfrak{l}$ is semisimple).

Then, from Proposition 3.3, the homogeneous manifold G/K admits a cyclic metric adapted to the transversally symmetric fibration $\pi\colon G/K \to G/L$ and the fibre type $F = L/K$ is isotropy-irreducible. Since $(\mathfrak{l}, s)$ satisfies the hypothesis of Lemma 3.5, we can give the next

Definition 3.6. We say that the cyclic homogeneous Riemannian manifold G/K has *irreducible fibre of type* S1, S2, NS0, NS1 *or* NS2 if $(\mathfrak{l}, s)$ is of that type.

All these cyclic homogeneous Riemannian manifolds G/K with irreducible fibre are listed as the pairs (G, K) on the first and third columns of Tables 1 and 2 or the corresponding pairs of Lie algebras $(\mathfrak{g}, \mathfrak{k})$ in Table 3. Note that all the connected subgroups K in Tables 1 and 2 are compact.

As for the simply connectedness of the manifolds L/K (equivalenty, G/K) in Table 1 for the non-semisimple Lie groups $L = U(n)$ and $L = S(U(p) \times U(q))$ when L/K is not a circle, we give the next

Remark 3.7. Consider the universal covering maps $\widetilde{\pi}\colon \widetilde{L} = \mathbb{R} \times SU(n) \to L = U(n)$ and $\widetilde{\rho}\colon \widetilde{L} \to U(1) \times SU(n)$, given by $\widetilde{\pi}(t, A) = \mathrm{e}^{\mathrm{i}t} A$ and $\widetilde{\rho}(t, A) = (\mathrm{e}^{\mathrm{i}t}, A)$, and the covering map $\rho\colon (\mathrm{e}^{\mathrm{i}t}, A) \in U(1) \times SU(n) \to \mathrm{e}^{\mathrm{i}t} A \in U(n)$, so that $\rho \circ \widetilde{\rho} = \widetilde{\pi}$. The subgroup of $U(n)$ defined by the maximal Lie subalgebra $\mathfrak{u}(1) \oplus \mathfrak{so}(n)$ of $\mathfrak{u}(1) \oplus \mathfrak{su}(n)$ is the compact group $K = \rho(U(1) \times SO(n))$. Now,

$$\operatorname{Ker} \widetilde{\pi}_{|\mathbb{R} \times SO(n)} = \begin{cases} \{(l\pi, \mathrm{e}^{-\mathrm{i}l\pi}) : l \in \mathbb{Z}\} & \text{if } n \text{ even} \\ \{(2l\pi, \mathrm{e}^{-\mathrm{i}2l\pi}) : l \in \mathbb{Z}\} & \text{if } n \text{ odd,} \end{cases}$$

then

$$\operatorname{Ker} \rho_{|U(1) \times SO(n)} = \widetilde{\rho}(\operatorname{Ker} \widetilde{\pi}_{|\mathbb{R} \times SO(n)}) = \begin{cases} \{(1, I_n), (-1, -I_n)\} & \text{if } n \text{ even} \\ \{(1, I_n)\} & \text{if } n \text{ odd,} \end{cases}$$

that is,

$$K = \rho(U(1) \times SO(n)) = \begin{cases} (U(1) \times SO(n))/\mathbb{Z}_2 & \text{if } n \text{ even} \\ U(1) \times SO(n) & \text{if } n \text{ odd.} \end{cases}$$

If $\widetilde{K} = \widetilde{\pi}^{-1}(K)$, then L/K and $\widetilde{L}/\widetilde{K}$ are diffeomorphic, and since $\widetilde{L}$ is connected and simply connected, the cardinal of $\pi_1(\widetilde{L}/\widetilde{K})$ coincides with the number of connected components of $\widetilde{K}$. Now, it is easy to see that

$$\widetilde{K} = \big(\mathbb{R} \times SO(n)\big) \cup \big(\mathbb{R} \times \mathrm{e}^{\mathrm{i}2\pi/n} SO(n)\big) \cup \cdots \cup \big(\mathbb{R} \times \mathrm{e}^{\mathrm{i}2(n-1)\pi/n} SO(n)\big),$$

and $\widetilde{K}$ has either n connected components (if n is odd) or $n/2$ connected components (if n is even). In particular, $L/K = U(n)/\rho(U(1) \times SO(n))$ is simply connected if and only if $n = 2$. Analogously, one obtains that $U(n)/((U(1) \times Sp(n/2)/\mathbb{Z}_2)$ is not simply connected for n even $\geqslant 4$. On the other hand, the complex Grassmannian $U(n)/(U(i) \times U(j)) \approx SU(n)/S(U(i) \times U(j))$ is simply connected, and it is a homogeneous spin Riemannian symmetric spin space for $n = i + j$ even [4].

In a similar way, if $L = S(U(p) \times U(q))$, $p \geqslant q \geqslant 1$, we consider its universal cover $\widetilde{L} = \mathbb{R} \times SU(p) \times SU(q)$ under the map $\widetilde{\pi}\colon \widetilde{L} \to L$ given by $\widetilde{\pi}(t, A, B) = \operatorname{diag}(\mathrm{e}^{\mathrm{i}qt}A, \mathrm{e}^{-\mathrm{i}pt}B)$, which defines the covering map

$$\rho\colon (\mathrm{e}^{\mathrm{i}t}, A, B) \in U(1) \times SU(p) \times SU(q) \to \begin{pmatrix} \mathrm{e}^{\mathrm{i}qt}A & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}pt}B \end{pmatrix} \in L,$$

and this can be used to obtain the (compact) subgroups, among others, of L defined by the maximal compact subalgebras $\mathfrak{u}(1) \oplus \mathfrak{su}(p) \oplus \mathfrak{so}(q)$, $\mathfrak{u}(1) \oplus \mathfrak{su}(p) \oplus \mathfrak{sp}(q/2)$ (for q even) and $\mathfrak{u}(1) \oplus \Delta\mathfrak{su}(p)$ (for $p = q$) of $\mathfrak{u}(1) \oplus \mathfrak{su}(p) \oplus \mathfrak{su}(q)$. We can also see via the nonconnectedness of $\widetilde{K} = \widetilde{\pi}^{-1}(K)$ that if $K = \rho(U(1) \times SU(p) \times SO(q))$ (for $q \geqslant 3$), or $\rho(U(1) \times SU(p) \times Sp(q/2))$ (for q even $\geqslant 4$), or $\rho(U(1) \times \Delta SU(p))$ (for $p = q \geqslant 3$), then L/K is not simply connected. Moreover, if $K = S(U(i) \times U(j) \times U(k))$ (for $i + j = p$, $k = q$, or $i = p$, $j + k = q$), then $\widetilde{\pi}^{-1}(K)$ is the connected subgroup $\mathbb{R} \times S(U(i) \times U(j)) \times SU(q)$ or $\mathbb{R} \times SU(p) \times S(U(j) \times U(k))$ of $\widetilde{L}$ and so in this case L/K is simply connected.

4. Homogeneous spin structures on transversally symmetric fibrations of noncompact type

Let $\xi = (E, \pi, B; F)$ be a fibration and let τ_F denote the bundle along the fibres of ξ. The quotient of the tangent bundle $T(E)$ of the total space E of ξ by τ_F is equivalent [3, Proposition 7.6] to the fibration induced by π from the tangent bundle of the base space B, so $T(E)$ decomposes as the Whitney sum

$$T(E) = \tau_F \oplus \pi^* T(B) \tag{4.1}$$

and hence the corresponding total Stiefel-Whitney classes satisfy $w(T(E)) = w(\tau_F)\, w(\pi^* T(B))$. Thus,

$$\begin{aligned} w_1(T(E)) &= w_1(\tau_F) + w_1(\pi^* T(B)), \\ w_2(T(E)) &= w_2(\tau_F) + w_1(\tau_F) w_1(\pi^* T(B)) + w_2(\pi^* T(B)). \end{aligned} \tag{4.2}$$

In our case, the homogeneous fibration $L/K \overset{i}{\hookrightarrow} G/K \overset{\pi}{\to} G/L$ induces the fibration $T(L/K) \overset{i_*}{\hookrightarrow} T(G/K) \overset{\pi_*}{\to} T(G/L)$ and we have the following result.

Proposition 4.1. *Let $\pi\colon M = G/K \to N = G/L$ be a transversally symmetric fibration of noncompact type. Then we have:*

(i) *M is a spin manifold if and only if the fibre type $F = L/K$ is so*;
(ii) *if F is simply connected then M is a homogeneous spin Riemannian manifold if and only if F is also a homogeneous spin Riemannian manifold.*

Proof. (i) We have $w_1(G/L) = 0$ and $w_2(G/L) = 0$. Furthermore, formula (4.1) is here

$$T(G/K) = i_*(T(L/K)) \oplus \pi^*(T(G/L))$$

and $w_1\big(\pi^*T(G/L)\big) = 0$, $w_2\big(\pi^*T(G/L)\big) = 0$. It follows from (4.2) that $w_1(G/K) = 0$ if and only if $w_1(L/K) = 0$, and $w_2(G/K) = 0$ if and only if $w_2(L/K) = 0$.

(ii) follows by using Remark 3.4, Proposition 2.4 and (i). □

Next, we show Theorem 1.3, which generalizes Proposition 4.1,(ii) when the fibre type F is not necessarily simply connected.

Proof of Theorem 1.3. From Proposition 2.4 and Remark 2.5, N is a homogeneous spin Riemannian manifold, $\widetilde{G}/\widetilde{L}$ is an adapted quotient expression and $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ is an adapted reductive decomposition.

Suppose that $(M = G/K, g)$ is a homogeneous spin Riemannian manifold, where g is a metric adapted to the homogeneous fibration and $\widetilde{G}/\widetilde{K}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ are an adapted quotient expression and an adapted reductive decomposition for the homogeneous spin structure. Since $\mathfrak{m}$ and $\mathfrak{p}$ are $\mathrm{Ad}^{\widetilde{G}}(\widetilde{K})$-invariant subspaces of $\mathfrak{g}$ and $\mathfrak{f}$ and $\mathfrak{p}$ are B-orthogonal in $\mathfrak{m}$, we have that $\mathfrak{f}$ must also be $\mathrm{Ad}^{\widetilde{G}}(\widetilde{K})$-invariant. Moreover, the restriction $\langle\cdot,\cdot\rangle_{\mathfrak{f}}$ to $\mathfrak{f}$ of the $\mathrm{Ad}^{\widetilde{G}}(\widetilde{K})$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$ which gives the adapted metric g on M, is also $\mathrm{Ad}^{\widetilde{G}}(\widetilde{K})$-invariant. Applying Lemma 2.3, F can be expressed as the homogeneous Riemannian manifold $\widetilde{L}/\widetilde{K}$. From the exact homotopy sequence of the fibration $\widetilde{L} \to \widetilde{G} \to N = \widetilde{G}/\widetilde{L}$,

$$\cdots \to \pi_2(N) \to \pi_1(\widetilde{L}) \to \pi_1(\widetilde{G}) \to \pi_1(N) \to \pi_0(\widetilde{L}) \to \pi_0(\widetilde{G}),$$

one has that $\widetilde{L}$ is connected and simply connected. Then, since from Proposition 4.1(i), F admits a spin structure, the result follows from Lemma 2.2.

Next, we shall prove the converse. Because $\mathfrak{f}$ and $\mathfrak{p}$ are $\mathrm{Ad}^{\widetilde{G}}(\widetilde{K})$-invariant subspaces of $\mathfrak{g}$, one gets that $\mathfrak{m} = \mathfrak{f} \oplus \mathfrak{p}$ is an $\mathrm{Ad}^{\widetilde{G}}(\widetilde{K})$-invariant decomposition of $\mathfrak{m}$. We consider on $\mathfrak{m}$ the inner product $\langle\cdot,\cdot\rangle$ such that $\langle\mathfrak{f},\mathfrak{p}\rangle = 0$. This inner product coincides on $\mathfrak{p}$ with the $\mathrm{Ad}^{\widetilde{G}}(\widetilde{L})$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$, which determines the $\widetilde{G}$-invariant metric on N, and on $\mathfrak{f}$ with the $\mathrm{Ad}^{\widetilde{G}}(\widetilde{K})$-invariant inner product which determines the $\widetilde{L}$-invariant metric on F. Then $\langle\cdot,\cdot\rangle$ is $\mathrm{Ad}^{\widetilde{G}}(\widetilde{K})$-invariant and Lemma 2.3 implies that (M, g), g being the adapted G-invariant metric determined by $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$, is isometric to $(\widetilde{G}/\widetilde{K}, \tilde{g})$.

From Proposition 4.1,(i), M is a spin manifold. Then, using Lemma 2.2, (M, g) is in fact a homogeneous spin Riemannian manifold with adapted quotient expression $\widetilde{G}/\widetilde{K}$ and reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. The last part of the theorem is a direct consequence of Theorem 1.1 and Proposition 3.3. □

Applying Theorem 1.3 for $F = S^1$, we have

Corollary 4.2. *The total space $M = G/K$ of a transversally symmetric fibration of noncompact type and with irreducible symmetric fibre of type* NS0 *is a homogeneous spin Riemannian manifold with the simplest Dirac operator.*

5. Example: The nonsymmetric traceless cyclic homogeneous spin Riemannian manifold $\widetilde{SL(2,\mathbb{R})}$

The universal covering $\widetilde{SL(2,\mathbb{R})}$ of the Lie group $SL(2,\mathbb{R})$ has nonsymmetric related pair $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}(2,\mathbb{R}), 0)$. This corresponds to the fifth manifold in Table 1,

for $p = q = 1$, via the isomorphism between $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(1,1)$, the latter being isomorphic to the nonsymmetric pair $(\mathfrak{su}(1,1), \mathfrak{su}(1) \oplus \mathfrak{su}(1)) = (\mathfrak{su}(1,1), 0)$.

Consider the basis of $\mathfrak{sl}(2,\mathbb{R})$ given by

$$X_1 = a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_2 = b\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_3 = \frac{ab}{\sqrt{a^2+b^2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for $a, b > 0$, which is orthogonal with respect to the Killing form of $\mathfrak{sl}(2,\mathbb{R})$ and satisfies

$$[X_2, X_3] = rX_1, \quad [X_3, X_1] = sX_2, \quad [X_1, X_2] = tX_3,$$

where $r = 2b^2/\sqrt{a^2+b^2}$, $s = 2a^2/\sqrt{a^2+b^2}$, $t = -2\sqrt{a^2+b^2}$, so that $r, s > 0$ and $r + s + t = 0$.

Hence each left-invariant Riemannian metric $\langle\cdot,\cdot\rangle_{a,b}$ on $\widetilde{SL(2,\mathbb{R})}$, corresponding to the invariant inner product on $\mathfrak{sl}(2,\mathbb{R})$ making $\{X_1, X_2, X_3\}$ orthonormal for such a given pair $a = \frac{1}{2}\sqrt{t(r+t)} > 0$, $b = \frac{1}{2}\sqrt{t(s+t)} > 0$, is cyclic. Moreover, it is traceless, as $\sum_{j=1}^{3}\langle [X_j, X_i], X_j\rangle_{a,b} = 0$, $i = 1, 2, 3$.

The manifold $\widetilde{SL(2,\mathbb{R})}$ is a Lie group, hence parallelizable, so $(\widetilde{SL(2,\mathbb{R})}, \langle\cdot,\cdot\rangle_{a,b})$ is a spin Riemannian manifold, for each couple (a, b). Moreover, since it is a traceless cyclic homogeneous Riemannian manifold, it has Dirac operator like that on a Riemannian symmetric space. We now check this with more detail.

Denote by $\bar{X}_i$ the left-invariant vector fields on $\widetilde{SL(2,\mathbb{R})}$ defined by the elements X_i. The Koszul formula for left-invariant vector fields $\bar{X}, \bar{Y}, \bar{Z}$,

$$2\langle \nabla_{\bar{X}}\bar{Y}, \bar{Z}\rangle = \langle [\bar{X}, \bar{Y}], \bar{Z}\rangle - \langle [\bar{Y}, \bar{Z}], \bar{X}\rangle + \langle [\bar{Z}, \bar{X}], \bar{Y}\rangle,$$

gives, applying the property $r+s+t = 0$, the components $\nabla_{\bar{X}_i}\bar{X}_j$ of the Levi-Civita connection of $\langle\cdot,\cdot\rangle_{a,b}$,

$$\begin{array}{lll} \nabla_{\bar{X}_1}\bar{X}_1 = 0, & \nabla_{\bar{X}_1}\bar{X}_2 = -r\bar{X}_3, & \nabla_{\bar{X}_1}\bar{X}_3 = r\bar{X}_2, \\ \nabla_{\bar{X}_2}\bar{X}_1 = s\bar{X}_3, & \nabla_{\bar{X}_2}\bar{X}_2 = 0, & \nabla_{\bar{X}_2}\bar{X}_3 = -s\bar{X}_1, \\ \nabla_{\bar{X}_3}\bar{X}_1 = -t\bar{X}_2, & \nabla_{\bar{X}_3}\bar{X}_2 = t\bar{X}_1, & \nabla_{\bar{X}_3}\bar{X}_3 = 0. \end{array}$$

The fibre of $\Sigma(\widetilde{SL(2,\mathbb{R})})$ is $\mathrm{Spin}(3) \cong SU(2)$. We have endomorphisms $\rho(X_i) \in \mathrm{End}(\Delta(\mathfrak{sl}(2,\mathbb{R}))) = \mathrm{End}(\mathbb{C}^2)$, $i = 1, 2, 3$, and there is a well-defined Clifford multiplication

$$\mathfrak{sl}(2,\mathbb{R}) \otimes \Sigma(\widetilde{SL(2,\mathbb{R})})_w \to \Sigma(\widetilde{SL(2,\mathbb{R})})_w, \quad X \otimes \psi \mapsto X\cdot\psi = \rho(X)\psi,$$

for all $w \in \widetilde{SL(2,\mathbb{R})}$ and $X \in \mathfrak{sl}(2,\mathbb{R})$. Since we have the relation $U\cdot V + V\cdot U = -2\langle U, V\rangle I$, for all $U, V \in \mathrm{Cl}_{\mathbb{C}}(\mathfrak{sl}(2,\mathbb{R}))$, and where I denotes the identity element, we can choose a basis $\{\psi_1, \psi_2\}$ of $\Delta(\mathfrak{sl}(2,\mathbb{R}))$ with respect to which one has

$$\rho(X_1) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad \rho(X_2) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \rho(X_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The spin connection ∇^{Σ} induced by the Levi-Civita connection ∇ in the fibration $\Sigma(\widetilde{SL(2,\mathbb{R})})$ is given, from (2.1), by

$$\nabla^{\Sigma}_{\bar{X}_i}\psi = \bar{X}_i(\psi) + \frac{1}{4}\sum_{j,k=1}^{3}\Gamma^k_{ij}\bar{X}_j\cdot\bar{X}_k\cdot\psi, \qquad i = 1, 2, 3,$$

that is, by

$$\nabla^{\Sigma}_{\bar{X}_1}\psi = \bar{X}_1(\psi) - \frac{r}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\psi, \quad \nabla^{\Sigma}_{\bar{X}_2}\psi = \bar{X}_2(\psi) - \frac{s}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\psi,$$

$$\nabla^{\Sigma}_{\bar{X}_3}\psi = \bar{X}_3(\psi) - \frac{t}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\psi,$$

so, from (2.2), the Dirac operator is given by

$$D\psi = \sum_{i=1}^{3} X_i \cdot X_i(\psi) - \frac{1}{2}(r+s+t)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\psi = \sum_{i=1}^{3} X_i \cdot X_i(\psi),$$

as expected.

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TABLE 1. Nonsymmetric cyclic homogeneous Riemannian manifolds G/K of a (necessarily noncompact) classical simple absolutely real Lie group G, where K is a closed connected subgroup of the maximal compact subgroup L of G and L/K is isotropy-irreducible. The manifolds G/K marked with $\star$ on the seventh column, satisfying moreover the conditions on the sixth column, if any, are examples of nonsymmetric homogeneous spin Riemannian manifolds with Dirac operator like that on a Riemannian symmetric spin space. (As for the type, see Definition 3.6.)

G	L	K		Type		Spin
$SL(n,\mathbb{R})$	$SO(n)$	$SO(i)\times SO(j)$	(a)	S1	$j=1$	$\star$
$(n\geqslant 3)$					or n even	
		$U(n/2)$	(b)	S1	$n\geqslant 6$	$\star$
$SU^*(2n)$	$Sp(n)$	$U(n)$		S1	n odd $\geqslant 3$	$\star$
$(n\geqslant 2)$		$Sp(i)\times Sp(j)$	(a)	S1		$\star$
$SU(p,q)$	$S(U(p)$	$SU(p)\times SU(q)$		NS0		$\star$
$(p\geqslant q\geqslant 1)$	$\times U(q))$	$S(U(i)\times U(j)\times U(k))$	(c)	NS2		
		$(U(1)\times SO(p)\times SU(q))/\mathbb{Z}_q$	(d)	NS2		
		$(U(1)\times SO(p)\times SU(q))/\mathbb{Z}_{2q}$	(e)	NS2		
		$(U(1)\times SU(p)\times SO(q))/\mathbb{Z}_p$	(f)	NS2		
		$(U(1)\times SU(p)\times SO(q))/\mathbb{Z}_{2p}$	(g)	NS2		
		$(U(1)\times Sp(p/2)\times SU(q))/\mathbb{Z}_{2q}$	(e)	NS2		
		$(U(1)\times SU(p)\times Sp(q/2))/\mathbb{Z}_{2p}$	(g)	NS2		
		$U(1)\times \Delta SU(p)$	(h)	NS1		
		$(U(1)\times \Delta SU(p))/\mathbb{Z}_2$	(i)	NS1		
$SO_0(p,1)$	$SO(p)$	$SO(i)\times SO(j)$	(j)	S1	$j=1$	$\star$
$(p\geqslant 3)$					or p even	
		$U(p/2)$	(e)	S1	$p\geqslant 6$	$\star$
$SO_0(p,2)$	$SO(p)$	$SO(p)$		NS0		$\star$
$(p\geqslant 3)$	$\times SO(2)$	$SO(i)\times SO(j)\times SO(2)$	(j)	NS1	$j=1$	$\star$
					or p even	
		$U(p/2)\times SO(2)$	(e)	NS1	$p\geqslant 6$	$\star$
$SO_0(p,q)$	$SO(p)$	$SO(i)\times SO(j)\times SO(q)$	(j)	S2	$j=1$ or	$\star$
$(p\geqslant q\geqslant 3)$	$\times SO(q)$				p even	
		$SO(p)\times SO(j)\times SO(k)$	(k)	S2	$k=1$ or	$\star$
					q even	
		$U(p/2)\times SO(q)$	(e)	S2	$p\geqslant 6$	$\star$
		$SO(p)\times U(q/2)$	(g)	S2	$q\geqslant 6$	$\star$
		$\Delta SO(p)$	(l)	S1		$\star$
$SO^*(2n)$	$U(n)$	$U(1)\times SO(n)$	(m)	NS1		
$(n\geqslant 3)$		$(U(1)\times SO(n))/\mathbb{Z}_2$	(b)	NS1		
		$U(i)\times U(j)$	(a)	NS1	n even	$\star$
		$SU(n)$		NS0		$\star$
		$(U(1)\times Sp(n/2))/\mathbb{Z}_2$	(b)	NS1		
$Sp(n,\mathbb{R})$	$U(n)$	$U(1)\times SO(n)$	(m)	NS1		
$(n\geqslant 2)$		$(U(1)\times SO(n))/\mathbb{Z}_2$	(b)	NS1		
		$U(i)\times U(j)$	(a)	NS1	n even	$\star$
		$SU(n)$		NS0		$\star$
		$(U(1)\times Sp(n/2))/\mathbb{Z}_2$	(n)	NS1		
$Sp(p,q)$	$Sp(p)$	$U(p)\times Sp(q)$		S2	p odd $\geqslant 3$	$\star$
$(p\geqslant q\geqslant 1)$	$\times Sp(q)$	$Sp(p)\times U(q)$		S2	q odd $\geqslant 3$	$\star$
		$Sp(i)\times Sp(j)\times Sp(k)$	(c)	S2		$\star$
		$\Delta Sp(p)$	(l)	S1		$\star$

(a) $n=i+j$, $i\geqslant j\geqslant 1$; (b) n even; (c) either $i+j=p$, $i\geqslant j\geqslant 1$, $k=q$ or $i=p$, $j+k=q$, $j\geqslant k\geqslant 1$; (d) p odd; (e) p even; (f) q odd; (g) q even; (h) $p=q$ odd; (i) $p=q$ even; (j) $p=i+j$, $i\geqslant j\geqslant 1$; (k) $q=j+k$, $j\geqslant k\geqslant 1$; (l) $p=q$; (m) n odd; (n) n even $\geqslant 4$.

TABLE 2. Simply connected nonsymmetric cyclic homogeneous Riemannian manifolds G/K of a simple complex Lie group G (considered as a real Lie group), where K is a closed connected subgroup of the maximal compact subgroup L of G and L/K is isotropy-irreducible. *All of them are of type* S1 and are nonsymmetric homogeneous spin Riemannian manifolds with Dirac operator like that on a Riemannian symmetric spin space, whenever they satisfy moreover the necessary conditions on the fifth column, if any. The space marked with "no" is not spin.

G	L	K		Spin
$SL(n,\mathbb{C})$	$SU(n)$	$SO(n)$	(a)	n even
		$S(U(i)\times U(j))$	(b)	n even
		$Sp(n/2)$	(c)	
$SO(n,\mathbb{C})$	$SO(n)$	$SO(i)\times SO(j)$	(d)	j=1 or n even
		$U(n/2)$	(e)	
$Sp(n,\mathbb{C})$	$Sp(n)$	$U(n)$	(a)	n odd $\geqslant 3$
		$Sp(i)\times Sp(j)$	(f)	
$E_6^{\mathbb{C}}$	E_6	$Sp(4)/\mathbb{Z}_2$		
		$(SU(6)\times SU(2))/Z_2$		
		$(\mathrm{Spin}(10)\times SO(2))/\mathbb{Z}_4$		
		F_4		
$E_7^{\mathbb{C}}$	E_7	$SU(8)/\mathbb{Z}_2$		
		$(\mathrm{Spin}(12)\times SU(2))/\mathbb{Z}_2$		
		$(E_6\times U(1))/\mathbb{Z}_3$		
$E_8^{\mathbb{C}}$	E_8	$SO'(16)$	(g)	
		$(E_7\times SU(2))/\mathbb{Z}_2$		
$F_4^{\mathbb{C}}$	F_4	$(Sp(3)\times SU(2))/\mathbb{Z}_2$		**no**
		$\mathrm{Spin}(9)$		
$G_2^{\mathbb{C}}$	G_2	$(SU(2)\times SU(2))/\mathbb{Z}_2$		

(a) $n \geqslant 2$; (b) $i+j=n\geqslant 3$, $i\geqslant j\geqslant 1$; (c) n even, $n\geqslant 4$; (d) $i+j=n\geqslant 5$, $i\geqslant j\geqslant 1$; (e) n even, $n\geqslant 6$; (f) $i+j=n\geqslant 2$, $i\geqslant j\geqslant 1$. (g) see [15, §§8.2.11].

TABLE 3. Triples of Lie algebras $(\mathfrak{g}, \mathfrak{l}, \mathfrak{k})$ of Lie groups (G, L, K) such that G/K is a nonsymmetric cyclic homogeneous Riemannian manifold of a (necessarily noncompact) exceptional simple absolutely real Lie group G, where K is a closed connected subgroup of the maximal compact subgroup L of G and L/K is isotropy-irreducible. Except for the manifolds marked with "no" on the sixth column, the rest of manifolds G/K are examples of nonsymmetric homogeneous spin Riemannian manifolds with Dirac operator like that on a Riemannian symmetric spin space.

$\mathfrak{g}$	$\mathfrak{l}$	$\mathfrak{k}$		Type	Spin
$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(4)$	$\mathfrak{u}(4)$		S1	**no**
		$\mathfrak{sp}(i) \oplus \mathfrak{sp}(j)$	(a)	S1	
$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{u}(1)$		S2	
		$\mathfrak{so}(6) \oplus \mathfrak{su}(2)$		S2	
		$\mathfrak{s}(\mathfrak{u}(i) \oplus \mathfrak{u}(j)) \oplus \mathfrak{su}(2)$	(b)	S2	
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10) \oplus \mathfrak{so}(2)$	$\mathfrak{so}(i) \oplus \mathfrak{so}(j) \oplus \mathfrak{so}(2)$	(c)	NS1	
		$\mathfrak{u}(5) \oplus \mathfrak{so}(2)$		NS1	
		$\mathfrak{so}(10)$		NS0	
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_4$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$		S1	**no**
		$\mathfrak{so}(9)$		S1	
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(8)$	$\mathfrak{so}(8)$		S1	
		$\mathfrak{sp}(4)$		S1	
		$\mathfrak{s}(\mathfrak{u}(i) \oplus \mathfrak{u}(j))$	(d)	S1	
$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(12) \oplus \mathfrak{so}(2)$		S2	
		$\mathfrak{so}(i) \oplus \mathfrak{so}(j) \oplus \mathfrak{su}(2)$	(e)	S2	
		$\mathfrak{u}(6) \oplus \mathfrak{su}(2)$		S2	
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6 \oplus \mathfrak{so}(2)$	$\mathfrak{sp}(4) \oplus \mathfrak{so}(2)$		NS1	
		$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathfrak{so}(2)$		NS1	
		$\mathfrak{so}(10) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2)$		NS1	
		$\mathfrak{f}_4 \oplus \mathfrak{so}(2)$		NS1	
		$\mathfrak{e}_6$		NS0	
$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16)$	$\mathfrak{so}(i) \oplus \mathfrak{so}(j)$	(f)	S1	
		$\mathfrak{u}(8)$		S1	
$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{e}_7 \oplus \mathfrak{so}(2)$		S2	
		$\mathfrak{su}(8) \oplus \mathfrak{su}(2)$		S2	
		$\mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$		S2	
		$\mathfrak{e}_6 \oplus \mathfrak{so}(2) \oplus \mathfrak{su}(2)$		S2	
$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$	$\mathfrak{sp}(3) \oplus \mathfrak{so}(2)$		S2	
		$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$		S2	
		$\mathfrak{u}(3) \oplus \mathfrak{sp}(1)$		S2	
$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9)$	$\mathfrak{so}(i) \oplus \mathfrak{so}(j)$	(g)	S1	**no**
		$\mathfrak{so}(8)$		S1	
$\mathfrak{g}_{2(2)}$	$\mathfrak{so}(4)$	$\mathfrak{u}(2)$		S1	
		$\mathfrak{so}(3)$		S1	
		$\mathfrak{so}(2) \oplus \mathfrak{so}(2)$		S1	

(a) $i+j=4$, $i \geqslant j \geqslant 1$; (b) $i+j=6$, $i \geqslant j \geqslant 1$; (c) $i+j=10$, $i \geqslant j \geqslant 1$; (d) $i+j=8$, $i \geqslant j \geqslant 1$; (e) $i+j=12$, $i \geqslant j \geqslant 1$ (f) $i+j=16$, $i \geqslant j \geqslant 1$; (g) $i+j=9$, $i \geqslant j \geqslant 2$.

TABLE 4. Cahen-Gutt spaces (compact simply connected Riemannian symmetric spin spaces L/K with L simple).

L	K	
$SU(n)$	$SO(n)$	n even $\geqslant 4$
$SU(p+q)$	$S(U(p) \times U(q))$	$p+q$ even, $p \geqslant q \geqslant 1$
$SU(2n)$	$Sp(n)$	$n \geqslant 2$
$SO(n)$	$SO(n-1)$	$n \geqslant 3$
$SO(p+q)$	$SO(p) \times SO(q)$	$p+q$ even, $p \geqslant q \geqslant 2$
$SO(2n)$	$U(n)$	$n \geqslant 3$
$Sp(n)$	$U(n)$	n odd $\geqslant 3$
$Sp(p+q)$	$Sp(p) \times Sp(q)$	$p \geqslant q \geqslant 1$
E_6	$Sp(4)/\mathbb{Z}_2$ $(SU(6) \times SU(2))/\mathbb{Z}_2$ $(\mathrm{Spin}(10) \times SO(2))/\mathbb{Z}_4$ F_4	
E_7	$SU(8)/\mathbb{Z}_2$ $(\mathrm{Spin}(12) \times SU(2))/\mathbb{Z}_2$ $(E_6 \times U(1))/\mathbb{Z}_3$	
E_8	$SO'(16)$ $(E_7 \times SU(2))/\mathbb{Z}_2$	
F_4	$\mathrm{Spin}(9)$	
G_2	$(SU(2) \times SU(2))/\mathbb{Z}_2$	